2.3: Acceleration-Velocity Models

In Section 1.2 we discussed vertical motion of a mass m near the surface of the earth. If we neglect air resistance, then Newton's second law of motion ($F = ma$) implies that the velocity of the mass satisfies the equation

$$m\frac{dv}{dt} = F_G \tag{1}$$

where $F_G = -mg$ is the (downward-directed) force of gravity (9.8 m/s^2 or 32 ft/s^2).

Exercise 1. Suppose that a crossbow bolt is shot straight upward from the ground ($y_0 = 0$) with initial velocity $v_0 = 4.9$ (m/s). Find the maximum height of the bolt and the time it will spend aloft.

One thing that was not taken into consideration in (1) is the effect of air resistance. Adjusting for this we get

$$m\frac{dv}{dt} = F_G + F_R. \tag{2}$$

Newton showed in his *Principia Mathematica* that certain physical assumptions imply that F_R is proportional to the square of the velocity; i.e. $F_R = kv^2$. However, empirical data shows that this is not quite right and that $F_R = kv^p$ for some $1 \leq p \leq 2$. Let us consider the extreme cases.

Example 1.(p=1) We suppose $F_R = -kv$ for some constant k. Then we can rewrite (2) as

$$m\frac{dv}{dt} = -kv - mg \quad \text{or} \quad \frac{dv}{dt} = -\rho v - g, \tag{3}$$

where $\rho = k/m > 0$. We can easily solve (3) for

$$v(t) = \left(v_0 + \frac{g}{\rho}\right) e^{-\rho t} - \frac{g}{\rho}.$$

What is the **terminal speed** of the object?

Exercise 2. Suppose that a crossbow bolt is shot straight upward with intial velocity $v_0 = 49$ m/s from ground level. But now assume that air resistance is taken into account with $\rho = 0.04$. Use (3) to find the maximum height and time aloft of the bolt. Compare with Exercise 1.

Example 2. (p=2) We suppose $F_R = \pm kv^2 = -kv|v|$. The we can rewrite (2) as

$$m\frac{dv}{dt} = \pm kv|v| - mg \quad \text{or} \quad \frac{dv}{dt} = -g - \rho v|v|, \tag{4}$$

where $\rho = k/m > 0$. Considering the two cases separately, we can easily solve to find

$$y(t) = y_0 + \frac{1}{\rho}\ln\left|\frac{\cos(C_1 - t\sqrt{\rho g})}{\cos C_1}\right| \quad \text{(Upward Motion)}$$

or

$$y(t) = y_0 - \frac{1}{\rho}\ln\left|\frac{\cosh(C_2 - t\sqrt{\rho g})}{\cosh C_2}\right|. \quad \text{(Downward Motion)}$$

Homework. 1-11, 19-25 (odd)